AN ENTROPY PROOF OF THE ARITHMETIC MEAN-GEOMETRIC MEAN INEQUALITY

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Abstract. Many proofs are known for the inequality between the arithmetic mean and the geometric mean. This note gives a new derivation, interpreting the means as final and initial values of entropy, and the inequality as the second law of thermodynamics.

Imagine bodies of mass $m_1, m_2, \ldots$ having the same specific heat c. Recall that specific heat is the amount of heat needed to raise the temperature, by one degree, of a unit mass of that material; it is measured in joule/(kelvin · kg). Materials that warm or cool easily have small c, those that are hard to warm or cool have large c. Write

$$p_i = \frac{m_i}{M}, \quad M = \sum_i m_i$$

The p's satisfy $0 \leqslant p_i \leqslant 1 \; \forall i$, $\sum_i p_i = 1$. Here is a fact of nature: if those bodies, from initial temperatures $T_1, T_2, \ldots$, are brought into thermal contact with one another, then they eventually settle (asymptote) to a common temperature

$$\langle T \rangle = \sum_i p_i T_i$$

When the ith body changes its temperature by $\mathrm{d}T_i$, the heat it receives is $cMp_i\mathrm{d}T_i$, and its entropy changes by

$$\mathrm{d}S_i = \frac{cMp_i\mathrm{d}T_i}{T_i} = cMp_i \mathrm{d}\log T_i$$

The first equal sign encodes the definition of entropy—the intake of heat divided by the absolute temperature. Let us run the process from start to finish. As the final temperature is $\langle T \rangle$ by the fact above,

$$S_i^{\text{finish}} - S_i^{\text{start}} = cM\big(p_i \log\langle T \rangle - p_i \log T_i\big)$$

Summing over all bodies,

$$\begin{aligned} \sum_i S_i^{\text{finish}} - \sum_i S_i^{\text{start}} &= cM\Big(\sum_i p_i \log\langle T \rangle - \sum_i \log T_i^{p_i}\Big) \\ &= cM\Big(\log\langle T \rangle - \log \prod_i T_i^{p_i}\Big) \end{aligned}$$

Now the *second law of thermodynamics* says that the total entropy of the system must increase: the left-hand side is nonnegative. Hence so is the right-hand side, which implies

$$p_1T_1 + p_2T_2 + \cdots \geqslant T_1^{p_1} \times T_2^{p_2} \times \cdots$$

The sharp case is when all the initial temperatures were equal already

$$T_1 = T_2 = \cdots$$

for this is the only case that involves no heat exchange.

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